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Soft gluons in logarithmic summations

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Abstract. We demonstrate that all the known single- and double-logarithm summations for a parton distribution function can be unified in the Collins–Soper resummation technique by applying soft approximations appropriate in different kinematic regions to real gluon emissions. Neglecting the gluon longitudinal momentum, we obtain the k_{T} (double-logarithm) resummation for two-scale QCD processes, and the Balitsky–Fadin–Kuraev–Lipatov (single-logarithm) equation for one-scale processes. Neglecting the transverse momentum, we obtain the threshold (double-logarithm) resummation for two-scale processes, and the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (single-logarithm) equation for one-scale processes. If we keep the longitudinal and transverse momenta simultaneously, we derive a unified resummation for a large Bjorken variable x, and a unified evolution equation appropriate for both intermediate and small x.

1 Introduction

It is known that radiative corrections in perturbative QCD (PQCD) produce large logarithms at each order of the coupling constant. Double logarithms appear in processes involving two scales, such as $\ln^2(p^+b)$ with p^+ the large longitudinal momentum of a parton and $1/b$ the small inverse impact parameter, where b is conjugate to the parton transverse momentum k_{T}. In the kinematic end-point region with large Bjorken variable x, one has $\ln^2(1/N)$ from the Mellin transformation of $\ln(1-x)/(1-x)_+$, for which the two scales are the large p^+ and the small infrared cutoff $(1-x)p^+$ for gluon emissions from a parton. Single logarithms are generated in processes involving only one scale, such as $\ln p^+$ and $\ln(1/x)$, for which the relevant scales are the large p^+ and the small xp^+, respectively. These logarithmic corrections to a parton distribution function have been summed to all orders by various methods: the k_{T} resummation for $\ln^2(p^+b)$ [1], the threshold resummation for $\ln^2(1/N)$ [2–4], the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) equation for $\ln p^+$ [5], and the Balitsky–Fadin–Kuraev–Lipatov (BFKL) equation for $\ln(1/x)$ [6].

In this paper we shall demonstrate that all the above single- and double-logarithm summations can be derived in the Collins–Soper (CS) resummation technique [1]. The main feature is the soft approximation for real gluon emissions, with which a parton distribution function $\phi(x + l^+/p^+, |\mathbf{k}_{\mathrm{T}} + \mathbf{l}_{\mathrm{T}}|)$ is associated. The arguments of ϕ indicate that the parton, emerging from a hadron, carries the longitudinal momentum $xp^+ + l^+$ and the transverse momentum $\mathbf{k}_{\mathrm{T}} + \mathbf{l}_{\mathrm{T}}$ in order to radiate a real gluon with momentum l. If we neglect the l^+ dependence,

$$\phi(x + l^+/p^+, |\mathbf{k}_{\mathrm{T}} + \mathbf{l}_{\mathrm{T}}|) \approx \phi(x, |\mathbf{k}_{\mathrm{T}} + \mathbf{l}_{\mathrm{T}}|), \qquad (1)$$

the scale $(1-x)p^+$ will not appear. Hence, (1) is inappropriate for the region with large $x \to 1$. In this soft approximation, we derive the k_{T} resummation for intermediate x, which involves two scales: the large xp^+ and the small k_{T}, and the BFKL equation for small x, which involves only one scale $xp^+ \approx k_{\mathrm{T}}$. If we neglect the l_{T} dependence,

$$\phi(x + l^+/p^+, |\mathbf{k}_{\mathrm{T}} + \mathbf{l}_{\mathrm{T}}|) \approx \phi(x + l^+/p^+, k_{\mathrm{T}}), \qquad (2)$$

the transverse degrees of freedom of a parton can be integrated out, and k_{T} will not be a relevant scale. Therefore, (2) is inappropriate for small x, where the scale k_{T} is of order xp^+, and not negligible. In this soft approximation, we derive the threshold resummation for large x, which involves two scales: the large xp^+ and the small $(1-x)p^+$, and the DGLAP equation for intermediate x, which involves only one scale, $xp^+ \sim (1-x)p^+$.

In the regions where (1) and (2) are inappropriate, we should keep both the l^+ and l_{T} dependences, and employ $\phi(x + l^+/p^+, |\mathbf{k}_{\mathrm{T}} + \mathbf{l}_{\mathrm{T}}|)$ for real gluon emissions directly. In this case the three scales xp^+, $(1-x)p^+$ and k_{T} exist simultaneously. We shall derive a unified resummation (a unification of the k_{T} and threshold resummations) for large x, and a unified evolution equation (a unification of the DGLAP and BFKL equations), which is suitable for both intermediate and small x. In conclusion, we are able to reproduce all the logarithmic summations and derive their unifications simply by employing appropriate soft approximations for real gluon emissions in the CS technique. The results are summarized in Table 1.

Table 1. Logarithmic summations derived from the Collins–Soper technique under different soft approximations at different Bjorken variables x

	Small x	Intermediate x	Large x
Neglect l^+	BFKL equation	k_{T} resummation	
Neglect l_{T}		DGLAP equation	threshold resummation
No neglect	unified	equation	unified resummation

2 Master equation

Consider a parton distribution function $\phi(x, k_{\mathrm{T}}, p^+)$ for a hadron with the light-like momentum $p^\mu = p^+\delta^{\mu+}$, which describes the probability that a parton carries the longitudinal momentum xp^+ and the transverse momentum k_{T}. If the parton is a quark, ϕ is written, in the minimal subtraction scheme, as

$$\phi(x, k_{\mathrm{T}}, p^+) = \int \frac{\mathrm{d}y^-}{2\pi} \int \frac{\mathrm{d}^2 y_{\mathrm{T}}}{4\pi^2} \mathrm{e}^{-\mathrm{i}xp^+y^- + \mathrm{i}\mathbf{k}_{\mathrm{T}}\cdot\mathbf{y}_{\mathrm{T}}} \times \langle p|\bar{q}(y^-, \mathbf{y}_{\mathrm{T}})\frac{1}{2}\gamma^+ q(0)|p\rangle, \quad (3)$$

where γ^+ is a Dirac matrix, and $|p\rangle$ denotes the hadron. Averages over spin and color are understood. If the parton is a gluon, the operator in the hadronic matrix element is replaced by $F^+_\mu(y^-, \mathbf{y}_{\mathrm{T}})F^{\mu+}(0)$. The above definition is given in the axial gauge $n \cdot A = 0$ with the gauge vector $n^\mu = \delta^{\mu-}$ lying on the light cone. To implement the CS technique, we allow n to vary arbitrarily away from the light cone ($n^2 \neq 0$) [1], and the parton distribution function becomes gauge dependent. However, it will be observed that the kernels for various logarithmic summations turn out to be n-independent. This is natural, since it has been shown that parton distribution functions defined for different n possess the same infrared structure, and thus the same evolution behavior, though they have a different ultraviolet structure [7]. After the derivation, we bring n back to the light cone, and the gauge invariance of the parton distribution function is restored. That is, the arbitrary vector n appears only at the intermediate stage of the derivation, and acts as an auxiliary tool.

The master equation in the CS technique is a differential equation of ϕ in p^+ [1,8]. Because of the scale invariance of ϕ in n as indicated by the gluon propagator, $-\mathrm{i}N^{\mu\nu}(l)/l^2$, with

$$N^{\mu\nu} = g^{\mu\nu} - \frac{n^\mu l^\nu + n^\nu l^\mu}{n \cdot l} + n^2 \frac{l^\mu l^\nu}{(n \cdot l)^2}, \quad (4)$$

ϕ depends on p^+ via the ratio $(p \cdot n)^2/n^2$. Hence, we have the chain rule relating the derivative $\mathrm{d}\phi/\mathrm{d}p^+$ to $\mathrm{d}\phi/\mathrm{d}n_\alpha$,

$$p^+ \frac{\mathrm{d}}{\mathrm{d}p^+}\phi = -\frac{n^2}{v \cdot n} v_\alpha \frac{\mathrm{d}}{\mathrm{d}n_\alpha}\phi, \quad (5)$$

with v a dimensionless vector along p. The operator $\mathrm{d}/\mathrm{d}n_\alpha$ applies to gluon propagators, leading to

$$\frac{\mathrm{d}}{\mathrm{d}n_\alpha} N^{\mu\nu} = -\frac{1}{n \cdot l}(l^\mu N^{\alpha\nu} + l^\nu N^{\mu\alpha}). \quad (6)$$

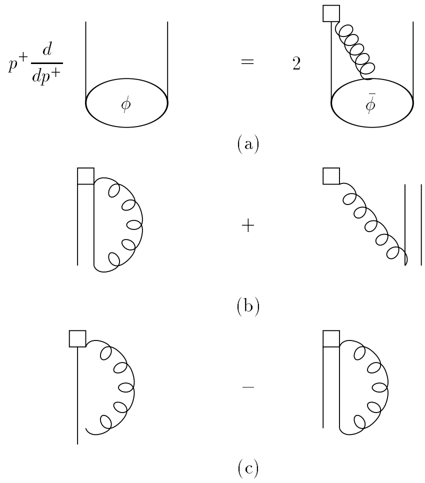

Fig. 1. a The derivative $p^+\mathrm{d}\phi/\mathrm{d}p^+$ in the axial gauge. **b** The soft structure and **c** the ultraviolet structure of the $O(\alpha_{\mathrm{s}})$ subdiagram containing the special vertex

The loop momentum l^μ (l^ν) contracts with the vertex the differentiated gluon attachments, which is then replaced by a special vertex,

$$\hat{v}_\alpha = \frac{n^2 v_\alpha}{v \cdot n n \cdot l}. \quad (7)$$

This special vertex can be read off from the combination of (5) and (6).

Employing Ward identities [9], a diagram with the contraction of l^μ can be expressed as the difference of a diagram in which the particle (quark or gluon) propagator after the contraction is removed, and a diagram in which the particle propagator before the contraction is removed. Hence, pair cancellation exists between the diagrams with adjacent contractions of l^μ. The summation of all the diagrams with different differentiated gluons then reduces to a single new diagram, where the external particle propagator lying farthest away is removed. That is, the special vertex appears at the outer end of the parton line in this new diagram. We obtain the master equation [1,8],

$$p^+ \frac{\mathrm{d}}{\mathrm{d}p^+}\phi(x, k_{\mathrm{T}}, p^+) = 2\bar{\phi}(x, k_{\mathrm{T}}, p^+), \quad (8)$$

shown in Fig. 1a, where the new diagram, denoted by $\bar{\phi}$, contains the special vertex represented by a square. The coefficient 2 comes from the equality of $\bar{\phi}$ with the special vertex on either of the two parton lines.

The collinear region of the loop momentum l is not important because of the factor $1/(n\cdot l)$ in $\hat{v}_\alpha$ with non-vanishing n^2. Therefore, the important regions of l are the soft and hard ones, in which the subdiagram containing the special vertex can be factorized by $\bar{\phi}$ according to Figs. 1b and c at lowest order, respectively. The second subdiagram in Fig. 1c, as a soft subtraction, guarantees a hard momentum flow. The remaining part is the original distribution function ϕ. Therefore, $\bar{\phi}$ is factorized into the convolution of the subdiagram containing the special vertex with ϕ.

The soft contribution from Fig. 1b is written as

$$\bar{\phi}_{\rm s}(x,k_{\rm T},p^+)=\bar{\phi}_{\rm sv}(x,k_{\rm T},p^+)+\bar{\phi}_{\rm sr}(x,k_{\rm T},p^+), \tag{9}$$

with

$$\bar{\phi}_{\rm sv}=\left[{\rm i}g^2C_F\mu^\epsilon\int\frac{{\rm d}^{4-\epsilon}l}{(2\pi)^{4-\epsilon}}N_{\nu\beta}(l)\frac{\hat{v}^\beta v^\nu}{v\cdot l l^2}-\delta K\right]\times\phi(x,k_{\rm T},p^+), \tag{10}$$

$$\bar{\phi}_{\rm sr}={\rm i}g^2C_F\mu^\epsilon\int\frac{{\rm d}^{4-\epsilon}l}{(2\pi)^{4-\epsilon}}N_{\nu\beta}(l)\frac{\hat{v}^\beta v^\nu}{v\cdot l}2\pi{\rm i}\delta(l^2)\times\phi(x+l^+/p^+,|\mathbf{k}_{\rm T}+\mathbf{l}_{\rm T}|,p^+), \tag{11}$$

corresponding to the virtual and real gluon emissions, respectively. The color factor $C_F=4/3$ should be replaced by $N_{\rm c}=3$ in the case that the parton is a gluon. The additive counterterm δK is specified in the modified minimal subtraction scheme. The hard contribution from Fig. 1c is given by

$$\bar{\phi}_h(x,k_{\rm T},p^+)=G(xp^+/\mu,\alpha_{\rm s}(\mu))\phi(x,k_{\rm T},p^+), \tag{12}$$

with the hard function

$$\begin{aligned}G&=-{\rm i}g^2C_F\mu^\epsilon\int\frac{{\rm d}^{4-\epsilon}l}{(2\pi)^{4-\epsilon}}N_{\nu\beta}(l)\frac{\hat{v}^\beta}{l^2}\\&\quad\times\left[\frac{x\not p-\not l}{(xp-l)^2}\gamma^\nu+\frac{v^\nu}{v\cdot l}\right]-\delta G\\&=-\frac{\alpha_{\rm s}(\mu)}{\pi}C_F\ln\frac{xp^+\nu}{\mu},\end{aligned} \tag{13}$$

where δG is an additive counterterm. In the case that the parton is a gluon, the expression of G can be written down straightforwardly. The gauge factor $\nu=((v\cdot n)^2/|n^2|)^{1/2}$ confirms our argument that ϕ depends on p^+ via the ratio $(p\cdot n)^2/n^2$.

3 $k_{\rm T}$ resummation and BFKL equation

We first discuss the soft approximation in (1) for ϕ associated with the real gluon emission. Fourier transforming (11) into the impact parameter b space in order to decouple the $l_{\rm T}$ integration, we derive

$$\bar{\phi}_{\rm s}(x,b,p^+)=K(1/(b\mu),\alpha_{\rm s}(\mu))\phi(x,b,p^+), \tag{14}$$

with the soft function

$$\begin{aligned}K&={\rm i}g^2C_F\mu^\epsilon\int\frac{{\rm d}^{4-\epsilon}l}{(2\pi)^{4-\epsilon}}N_{\nu\beta}(l)\frac{\hat{v}^\beta v^\nu}{v\cdot l}\\&\quad\times\left[\frac{1}{l^2}+2\pi{\rm i}\delta(l^2){\rm e}^{{\rm i}\mathbf{l}_{\rm T}\cdot\mathbf{b}}\right]-\delta K\\&=\frac{\alpha_{\rm s}(\mu)}{\pi}C_F\ln\frac{1}{b\mu}.\end{aligned} \tag{15}$$

Hence, in the intermediate x region ϕ involves two scales: the large xp^+ that characterizes the hard function G in (13) and the small $1/b$ that characterizes the soft function K.

Using $\bar{\phi}=\bar{\phi}_{\rm s}+\bar{\phi}_h$, the master equation (8) becomes

$$\begin{aligned}&p^+\frac{\rm d}{{\rm d}p^+}\phi(x,b,p^+)\\&=2\left[K(1/(b\mu),\alpha_{\rm s}(\mu))+G(xp^+/\mu,\alpha_{\rm s}(\mu))\right]\phi(x,b,p^+).\end{aligned} \tag{16}$$

Since both the ultraviolet divergences in K and G come from the virtual gluon contribution $\bar{\phi}_{\rm sv}$, they cancel each other such that $K+G$ is renormalization-group (RG) invariant. The single logarithms $\ln(b\mu)$ and $\ln(xp^+/\mu)$, contained in K and G, respectively, are organized by the RG equations

$$\mu\frac{\rm d}{{\rm d}\mu}K=-\gamma_K=-\mu\frac{\rm d}{{\rm d}\mu}G. \tag{17}$$

The anomalous dimension of K, $\lambda_K=\mu{\rm d}\delta K/{\rm d}\mu$, is given, up to two loops, by [10]

$$\gamma_K=\frac{\alpha_{\rm s}}{\pi}C_F+\left(\frac{\alpha_{\rm s}}{\pi}\right)^2C_F\left[C_A\left(\frac{67}{36}-\frac{\pi^2}{12}\right)-\frac{5}{18}n_f\right], \tag{18}$$

with $n_{\rm f}$ the number of quark flavors and $C_A=3$ a color factor. The solution of (17) gives

$$\begin{aligned}&K(1/(b\mu),\alpha_{\rm s}(\mu))+G(xp^+/\mu,\alpha_{\rm s}(\mu))\\&\quad=-\int_{1/b}^{xp^+}\frac{{\rm d}\mu}{\mu}\gamma_K(\alpha_{\rm s}(\mu)),\end{aligned} \tag{19}$$

where the initial conditions $K(1,\alpha_{\rm s}(1/b))$ and $G(1,\alpha_{\rm s}(xp^+))$ that contribute only to the single-logarithm summation have been dropped. Solving the differential equation (16) with the above expression inserted, we obtain the $k_{\rm T}$ resummation [8],

$$\phi(x,b,p^+)=\varDelta_k(b,xp^+)\phi^{(0)}(x), \tag{20}$$

with the (Sudakov) exponential

$$\varDelta_k(b,xp^+)=\exp\left[-2\int_{1/b}^{xp^+}\frac{{\rm d}p}{p}\int_{1/b}^{p}\frac{{\rm d}\mu}{\mu}\gamma_K(\alpha_{\rm s}(\mu))\right]. \tag{21}$$

In the small x region with $xp^+\sim k_{\rm T}$, or $xp^+\sim 1/b$ in the b space, the above two-scale case reduces to the one-scale case. The source of double logarithms, i.e., the integral containing γ_K in (19), is less important. Instead of applying the RG equation (17), we simply add (10)–(12), or equivalently, Figs. 1b and c. The result can be understood in the way that the function G introduces an ultraviolet cutoff of order $xp^+ \sim k_\mathrm{T}$, which comes from the first subdiagram of Fig. 1c, to the virtual soft gluon contribution. Without Fourier transformation, $\bar{\phi}$ can be re-expressed, according to (10) and (11), as

$$\begin{aligned}&\bar{\phi}(x, k_\mathrm{T}, p^+)\\ &= \mathrm{i}g^2 N_\mathrm{c} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} N_{\nu\beta}(l) \frac{\hat{v}^\beta v^\nu}{v\cdot l} \qquad (22)\\ &\times\left[\frac{\theta(k_\mathrm{T}^2 - l_\mathrm{T}^2)}{l^2}\phi(x, k_\mathrm{T}, p^+) + 2\pi\mathrm{i}\delta(l^2)\phi(x, |\mathbf{k}_\mathrm{T} + \mathbf{l}_\mathrm{T}|, p^+)\right],\end{aligned}$$

where the color factor has been replaced by N_c, because we consider the gluon distribution function in the small x region. The θ function, defining the ultraviolet cutoff k_T, is the consequence of the inclusion of G.

To make a variation in x via a variation in p^+, we assume a fixed parton momentum. This assumption is reasonable, since we are considering the one-scale case with $xp^+ \sim k_\mathrm{T}$. The momentum fraction x is then proportional to $1/p^+$, and we have [9]

$$p^+ \frac{\mathrm{d}}{\mathrm{d}p^+}\phi(x, k_\mathrm{T}, p^+) = -x\frac{\mathrm{d}}{\mathrm{d}x}\phi(x, k_\mathrm{T}, p^+). \qquad (23)$$

Performing the integrations over l^+ and l^- in (22) and using (23), the master equation (8) reduces to the BFKL equation [11],

$$\begin{aligned}\frac{\mathrm{d}\phi(x, k_\mathrm{T}, p^+)}{\mathrm{d}\ln(1/x)} &= \bar{\alpha}_\mathrm{s} \int \frac{\mathrm{d}^2 l_\mathrm{T}}{\pi l_\mathrm{T}^2}\left[\phi(x, |\mathbf{k}_\mathrm{T} + \mathbf{l}_\mathrm{T}|, p^+)\right.\\ &\left.\quad -\theta(k_\mathrm{T}^2 - l_\mathrm{T}^2)\phi(x, k_\mathrm{T}, p^+)\right], \qquad (24)\end{aligned}$$

with the coupling constant $\bar{\alpha}_\mathrm{s} = N_\mathrm{c}\alpha_\mathrm{s}/\pi$.

4 Threshold resummation and DGLAP equation

We next consider the soft approximation in (2). In this case the dependence on k_T can be integrated out from both sides of (10)–(12), and the scale $(1-x)p^+$ enters. We employ the Mellin transformation to bring $\bar{\phi}_\mathrm{sr}$ from the momentum fraction x space to the moment N space,

$$\bar{\phi}_\mathrm{sr}(N, p^+) = \int_0^1 \mathrm{d}x x^{N-1}\bar{\phi}_\mathrm{sr}(x, p^+), \qquad (25)$$

under which the l^+ integration decouples. Combined with the soft virtual contribution in (10), we derive

$$\bar{\phi}_\mathrm{s}(N, p^+) = K(p^+/(N\mu), \alpha_\mathrm{s}(\mu))\phi(N, p^+), \qquad (26)$$

with the soft function

$$\begin{aligned}K &= \mathrm{i}g^2 C_F \mu^\epsilon \int_0^1 \mathrm{d}z \int \frac{\mathrm{d}^{4-\epsilon} l}{(2\pi)^{4-\epsilon}} N_{\nu\beta}(l)\frac{\hat{v}^\beta v^\nu}{v\cdot l}\left[\frac{\delta(1-z)}{l^2}\right.\\ &\left.\quad + 2\pi\mathrm{i}\delta(l^2)\delta\left(1 - z - \frac{l^+}{p^+}\right)z^{N-1}\right] - \delta K,\\ &= \frac{\alpha_\mathrm{s}(\mu)}{\pi}C_F \ln\frac{p^+\nu}{N\mu}, \qquad (27)\end{aligned}$$

and the counterterm δK the same as that in (15). Therefore, in the large x region ϕ involves two scales, the large $xp^+ \sim p^+$ from the hard function G in (13) and the small $(1-x)p^+ \sim p^+/N$ from the soft function K.

Similarly, (16)–(19) hold but with $1/b$ replaced by p^+/N. To sum $\ln(1/N)$, we regard the derivative $p^+\mathrm{d}\phi/\mathrm{d}p^+$ as

$$p^+\frac{\mathrm{d}\phi}{\mathrm{d}p^+} = \frac{p^+}{N}\frac{\partial\phi}{\partial(p^+/N)}, \qquad (28)$$

which leads to the threshold resummation,

$$\phi(N, p^+) = \Delta_t(N, p^+)\phi^{(0)} \qquad (29)$$

with the exponential

$$\Delta_t(N, p^+) = \exp\left[-2\int_{p^+/N}^{p^+}\frac{\mathrm{d}p}{p}\int_p^{p^+}\frac{\mathrm{d}\mu}{\mu}\gamma_K(\alpha_\mathrm{s}(\mu))\right]. \qquad (30)$$

In the intermediate x region the above two-scale case reduces to a one-scale case because of $xp^+ \sim (1-x)p^+$, and the source of double logarithms becomes less important. Without the Mellin transformation, the addition of (10)–(12), with the soft approximation in (2) inserted, leads to the DGLAP equation [9],

$$p^+\frac{\mathrm{d}}{\mathrm{d}p^+}\phi(x, p^+) = \int_x^1 \frac{\mathrm{d}\xi}{\xi}P(x/\xi, p^+)\phi(\xi, p^+), \qquad (31)$$

with the kernel

$$P(z, p^+) = \frac{\alpha_\mathrm{s}(p^+)}{\pi}C_F\frac{2}{(1-z)_+}, \qquad (32)$$

where the variable change $\xi = x + l^+/p^+$ has been employed. The argument of α_s has been chosen as the single scale $xp^+ \sim (1-x)p^+$, which is of order p^+. Note that the kernel P differs from the splitting function [5]

$$P_{qq} = \frac{\alpha_\mathrm{s}}{\pi}C_F\frac{1+z^2}{(1-z)_+} \qquad (33)$$

by the term $(z^2-1)/(1-z)_+$, which is finite in the $z \to 1$ limit. The reason is that the real gluon emission is evaluated in the soft approximation on deriving P, while it is calculated exactly on deriving P_{qq}. Except P_{qq}, the other splitting functions P_{qg}, P_{gq} and P_{gg} can also be evaluated in our formalism. P_{gg} will be computed in the next section. For P_{qg}, the subdiagram containing the special vertex at the outer end of a quark line should be factorized in such a way that two gluons attach to it from below. For P_{gq}, the subdiagram containing the special vertex at the outer end of a gluon line is factorized in such a way that two quarks attach to it from below. Apparently, these subdiagrams start with $O(\alpha_\mathrm{s}^2)$. For example, the $g \to q\bar{q}$ cross section, responsible for P_{qg}, is of $O(\alpha_\mathrm{s})$ and the additional gluon emerging from the special vertex gives another α_s. Equation (32) indicates that we have reproduced only the most singular term $2/(1-z)_+$ of P_{qq} for the reason given in the previous paragraph. Since the leading-order P_{qg} and P_{gq}

do not possess such a singular term [5], the subdiagrams do not contribute to these splitting functions at $O(\alpha_s)$.

Another remark is in order. The BFKL equation is appropriate for the multi-Regge region, where the transverse momenta carried by the rung gluons of a ladder diagram are of the same order, i.e., $l_T \approx k_T$. Hence, the loop momentum l_T flowing through the gluon distribution function is not negligible. Since the gluon distribution function, rising fast at small x, is dominated by its behavior at $\xi \sim x$, we have the soft approximation in (1). While the DGLAP equation is appropriate for the transverse momentum ordered region, in which we have $l_T \ll p_T$, the soft approximation in (2) holds. The k_T dependence of the distribution function then decouples, and can be integrated out. This is the reason that a distribution function in the DGLAP equation does not need to involve the transverse degrees of freedom of a parton.

5 Unified logarithmic summations

In this section we study the case in which both the l^+ and l_T dependences of ϕ in (11) are retained. It will be shown that a unified resummation for large x and a unified evolution equation for intermediate and small x are derived. Obviously, we should apply both the Fourier and Mellin transformations to (11), and obtain

$$\begin{aligned}&\bar{\phi}_s(N,b,p^+)\\&\quad = K(p^+/(N\mu),1/(b\mu),\alpha_s(\mu))\phi(N,b,p^+), \end{aligned}\tag{34}$$

with the soft function

$$\begin{aligned}K &= \mathrm{i}g^2C_F\mu^\epsilon\int_0^1\mathrm{d}z\int\frac{\mathrm{d}^{4-\epsilon}l}{(2\pi)^{4-\epsilon}}N_{\nu\beta}(l)\frac{\hat{v}^\beta v^\nu}{v\cdot l}\left[\frac{\delta(1-z)}{l^2}\right.\\&\quad\left.+2\pi\mathrm{i}\delta(l^2)\delta\left(1-z-\frac{l^+}{p^+}\right)z^{N-1}\mathrm{e}^{\mathrm{i}\mathbf{l}_T\cdot\mathbf{b}}\right]-\delta K,\\&=\frac{\alpha_s(\mu)}{\pi}C_F\left[\ln\frac{1}{b\mu}-K_0\left(\frac{2\nu p^+b}{N}\right)\right],\end{aligned}\tag{35}$$

K_0 being the modified Bessel function. It is easy to examine the large b and N limits of the above expression: for $p^+b \gg N$, we have $K_0 \to 0$ and the soft function K approaches (15) for the k_T resummation. For $N \gg p^+b$, we have $K_0 \approx -\ln(\nu p^+b/N)$ and K approaches (27) for the threshold resummation.

Equation (35) implies a characteristic scale of order

$$\frac{1}{b}\exp\left[-K_0\left(\frac{p^+b}{N}\right)\right].\tag{36}$$

Following similar procedures as in (16)–(21) we derive the unified resummation,

$$\phi(N,b,p^+) = \Delta_u(N,b,p^+)\phi^{(0)},\tag{37}$$

with the exponential

$$\begin{aligned}&\Delta_u(N,b,p^+)\exp\left[-2\int_{\exp[-K_0(p^+b/N)]/b}^{p^+}\frac{\mathrm{d}p}{p}\right.\\&\quad\left.\times\int_{\exp[-K_0(p^+b)]/b}^{p}\frac{\mathrm{d}\mu}{\mu}\gamma_K(\alpha_s(\mu))\right],\end{aligned}\tag{38}$$

which is appropriate for arbitrary b and N. The lower bound of p corresponds to (36), while the lower bound of μ is motivated by the $b \to \infty$ and $b \to 0$ limits, at which (38) approaches (21) and (30), respectively.

The unified resummation for a k_T-dependent parton distribution function at the threshold exhibits suppression in the large b region ($p^+b \gg N$), and turns into enhancement in the small b region ($N \gg p^+b$). That is, (38) displays the opposite effects of the k_T and threshold resummations at different b. The behavior of the unified resummation can be explained as follows. For an intermediate x, virtual and real soft gluon corrections cancel exactly in the small b region, since they have an almost equal phase space. Hence, there are only single collinear logarithms, namely, no double logarithms. In this case the Sudakov exponential approaches unity as $b < 1/p^+$ [12], indicating the soft cancellation stated above. However, at threshold ($x \to 1$), real gluon emissions still do not have sufficient phase space even as $b \to 0$, and soft virtual corrections are not cancelled exactly. In this case the double logarithms $\ln^2(1/N)$ persist and become dominant. The Sudakov suppression then transits into an enhancement, instead of unity, and does so smoothly as b decreases.

In the intermediate and small x regions, it is not necessary to resum the double logarithms $\ln^2(1/N)$. After extracting the k_T resummation, the remaining single-logarithm summation corresponds to a unification of the DGLAP and BFKL equations, since both the l^+ and l_T dependences have been kept. We re-express the function ϕ in the integrand of $\bar{\phi}_{sr}$ under Fourier transformation as

$$\begin{aligned}&\phi(x+l^+/p^+,b,p^+)\\&\quad=\theta((1-x)p^+-l^+)\phi(x,b,p^+)+[\phi(x+l^+/p^+,b,p^+)\\&\quad\quad-\theta((1-x)p^+-l^+)\phi(x,b,p^+)].\end{aligned}\tag{39}$$

The contribution from the first term is combined with $\bar{\phi}_{sv}$, giving the soft function K for the k_T resummation. The RG solution of $K+G$ is given by

$$\begin{aligned}K+G &= \bar{\alpha}_s(xp^+)\left[\ln(1-x)+\ln(p^+b)\right]\\&\quad-\int_{1/b}^{xp^+}\frac{\mathrm{d}\mu}{\mu}\gamma_K(\alpha_s(\mu)),\end{aligned}\tag{40}$$

where the first term on the right-hand side comes from the extra θ function in (39). The color factor has been replaced by N_c, since we are considering the gluon distribution function. The contribution from the second term is written as

$$\begin{aligned}&\mathrm{i}N_cg^2\int\frac{\mathrm{d}^4l}{(2\pi)^4}N_{\nu\beta}(l)\frac{\hat{v}^\beta v^\nu}{v\cdot l}2\pi\mathrm{i}\delta(l^2)\mathrm{e}^{\mathrm{i}\mathbf{l}_T\cdot\mathbf{b}}\\&\times[\phi(x+l^+/p^+,b,p^+)-\theta((1-x)p^+-l^+)\phi(x,b,p^+)],\end{aligned}\tag{41}$$

which will generate the splitting function below.

The master equation (8) then becomes

$$\begin{aligned}&p^+\frac{\mathrm{d}}{\mathrm{d}p^+}\phi(x,b,p^+)\\&\quad=-2\left[\int_{1/b}^{xp^+}\frac{\mathrm{d}\mu}{\mu}\gamma_K(\alpha_s(\mu))-\bar{\alpha}_s(xp^+)\ln(p^+b)\right]\end{aligned}\tag{42}$$

$$\times\phi(x,b,p^+) + 2\bar{\alpha}_s(xp^+)\int_x^1 \mathrm{d}z P_{gg}(z)\phi(x/z,b,p^+),$$

with the splitting function

$$P_{gg} = \left[\frac{1}{(1-z)_+} + \frac{1}{z} - 2 + z(1-z)\right] \quad (43)$$

obtained from (41). The term $-2 + z(1-z)$, which is finite as $z \to 0$ and $z \to 1$, has been added. The term proportional to $\ln(1-x)$ in (40) has been absorbed into P_{gg} to arrive at the plus distribution $1/(1-z)_+$. We first extract the exponential Δ from the k_T resummation,

$$\Delta(x,b,Q_0,p^+) = \exp\left(-2\int_{xQ_0}^{xp^+}\frac{\mathrm{d}p}{p}\right. \quad (44)$$
$$\left.\times\left[\int_{1/b}^{p}\frac{\mathrm{d}\mu}{\mu}\gamma_K(\alpha_s(\mu)) - \bar{\alpha}_s(p)\ln\frac{pb}{x}\right]\right),$$

where Q_0 is an arbitrary low energy scale. It is easy to justify by substitution that the gluon distribution function is given by

$$\phi(x,b,Q)$$
$$= \Delta(x,b,Q_0,Q)\phi^{(0)} + 2\int_x^1 \mathrm{d}z\int_{Q_0}^{Q}\frac{\mathrm{d}\mu}{\mu}\bar{\alpha}_s(x\mu)$$
$$\times\Delta_k(x,b,\mu,Q)P_{gg}(z)\phi(x/z,b,\mu), \quad (45)$$

with $\phi^{(0)}$ the initial condition of ϕ. Equation (45) is the unified evolution equation, which can be regarded as a modified version of the Ciafaloni–Catani–Fiorani–Marchesini equation [13].

As mentioned above, the soft divergences from virtual and real radiative corrections cancel each other order by order. However, in the derivation of the CCFM equation virtual gluons are summed to all orders and grouped into the Sudakov form factor, while real gluon contributions are evaluated only to lowest order, which leads to the splitting function P_{gg} [13]. Therefore, the soft cancellation in the CCFM equation is not explicit. This may bring about a difficulty in the extension of the Sudakov resummation up to next-to-leading logarithms. In our approach the soft cancellation is fulfilled by (39), where the first term removes the soft pole in the virtual correction of the same order, and the second term, being infrared finite, gives the splitting function P_{gg}. With this clear partition of the real gluon contribution, we have derived the Sudakov form factor with the accuracy of next-to-leading logarithms.

6 Conclusion

In this paper we have demonstrated that all the known single- and double-logarithm summations, including their unifications, can be derived by the CS technique. The main feature is the treatment of the real gluon contributions to the subdiagram containing the special vertex. The derivations of the various logarithmic summations begin to diverge from this step. Simply adopting soft approximations appropriate in different kinematic regions, i.e., neglecting the l^+ or l_T dependence in the parton distribution function associated with the real gluon emission, the CS technique reduces to the k_T resummation, the BFKL equation, the threshold resummation, or the DGLAP equation. If keeping both the l^+ and l_T dependences, a unified resummation for large x and a unified evolution equation for intermediate and small x are obtained. Our conclusion has been summarized in Table 1.

We emphasize that the CS resummation technique must be applied to a QCD process after its factorization has been constructed. A QCD process usually involves more than one particle momentum. For example, the deep inelastic scattering (DIS) of a hadron involves the hadron momentum p and the virtual photon momentum q. Quark–quark scattering and annihilation amplitudes involve the two initial quark momenta p_1 and p_2 [14]. Since the CS technique starts with (5), which holds only in the case with a single argument $(p\cdot n)^2/n^2$, it cannot be applied to a QCD process directly. We must factorize a QCD process into several subprocesses, each of which depends only on a single momentum. For example, DIS is factorized into the convolution of a hard subamplitude, which depends only on the large momentum transfer $Q^2 \equiv -q^2$, with a parton distribution function, which depends only on the hadron momentum p as shown in (3). The CS technique is then applied to the parton distribution function following the steps presented in this work. A quark–quark scattering or annihilation amplitude can be factorized in a similar way, in which infrared divergences are absorbed into jet functions associated with energetic quarks. The CS technique is then applied to these jet functions [8], each of which depends only on a single quark momentum. The double logarithms that were resummed in [14] then appear as $\ln^2(p_1^{+2}/\mu^2) = \ln^2(p_2^{-2}/\mu^2) = \ln^2[s/(2\mu^2)]$, if p_1 and p_2 are chosen in the plus and minus directions, respectively, with $s = (p_1+p_2)^2$ being the center-of-mass energy.

Based on the master equation (8), many extensions can be performed. We have considered only the $O(\alpha_s)$ subdiagram, which in fact corresponds to the summation of ladder graphs, namely, the DGLAP and BFKL equations. As explained in Sect. 4, the neglect of the argument l_T in the distribution function associated with real gluon emissions corresponds to strong transverse momentum ordering. In some processes, such as quark–quark scattering mentioned above and the polarized structure function g_1, non-ladder graphs and the contribution from the region without the strong transverse momentum ordering are essential. For these processes, non-ladder graphs are included simply by evaluating the $O(\alpha_s^2)$ subdiagrams, which give next-to-leading-order corrections. While the contribution from the region without the transverse momentum ordering is taken into account by keeping the l_T dependence of the distribution function, similar to the derivation of the BFKL equation appropriate for the multi-Regge region.

The k_T and threshold resummations, and the DGLAP and BFKL equations have been widely studied and applied to many QCD processes. The new results obtained in this paper also have important applications. The unified resummation is appropriate for the analysis of the di-jet production [15], in which the transverse energy of one jet (the trigger jet) is measured, while the other jet (the probe jet) has a large rapidity up to 3.0, which corresponds to high x values. The unified evolution equation, because of its extra Q dependence at small x, is appropriate for the explanation of the HERA data of the proton structure function $F_2(x, Q^2)$ [16]. These subjects will be discussed elsewhere.

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References

1. J.C. Collins, D.E. Soper, Nucl. Phys. B **193**, 381 (1981)
2. G. Sterman, Phys. Lett. B **179**, 281 (1986); Nucl. Phys. B **281**, 310 (1987); E. Laenen, G. Oderda, G. Sterman, Report No. hep-ph/9806467
3. S. Catani, L. Trentadue, Nucl. Phys. B **327**, 323 (1989); Nucl. Phys. B **353**, 183 (1991)
4. G.P. Korchemsky, G. Marchesini, Nucl. Phys. B **406**, 225 (1993); Phys. Lett. B **313**, 433 (1993)
5. V.N. Gribov, L.N. Lipatov, Sov. J. Nucl. Phys. **15**, 438 (1972); G. Altarelli, G. Parisi, Nucl. Phys. B **126**, 298 (1977); Yu.L. Dokshitzer, Sov. Phys. JETP **46**, 641 (1977)
6. E.A. Kuraev, L.N. Lipatov, V.S. Fadin, Sov. Phys. JETP **45**, 199 (1977); Ya.Ya. Balitsky, L.N. Lipatov, Sov. J. Nucl. Phys. **28**, 822 (1978); L.N. Lipatov, Sov. Phys. JETP **63**, 904 (1986)
7. H-n. Li, Report No. hep-ph/9803202
8. H-n. Li, Phys. Lett. B **369**, 137 (1996); Phys. Rev. D **55**, 105 (1997)
9. H-n. Li, Chin. J. Phys. **37**, 8 (1999); Phys. Lett. B **405**, 347 (1997)
10. J. Botts, G. Sterman, Nucl. Phys. B **325**, 62 (1989)
11. J. Kwieciński, A.D. Martin, P.J. Sutton, Phys. Rev. D **53**, 6094 (1996).
12. H-n. Li, G. Sterman, Nucl. Phys. B **381**, 129 (1992)
13. M. Ciafaloni, Nucl. Phys. B **296**, 49 (1988); S. Catani, F. Fiorani, G. Marchesini, Phys. Lett. B **234**, 339 (1990); Nucl. Phys. B **336**, 18 (1990); G. Marchesini, Nucl. Phys. B **445**, 49 (1995)
14. R. Kirschner, L.N. Lipatov, Nucl. Phys. B **213**, 122 (1983)
15. J. Huston, Report No. hep-ph/9901352
16. ZEUS Collaboration, M. Derrick et al., Z. Phys. C **72**, 399 (1996); H1 Collaboration, S. Aid et al., Nucl. Phys. B **470**, 3 (1996)